

# Solution Behavior of the Transient Heat Transfer Problem in Thermoelectric Shape Memory Alloy Actuators

Zhonghai Ding\* Dimitris C. Lagoudas\*

## Abstract

The main purpose of this paper is to study the solution behavior of the transient heat transfer problem for one dimensional symmetric thermoelectric shape memory alloy (SMA) actuators. It is proved that for the transient cooling problem with constant electric current density of magnitude  $|J|$  there is a value  $J_0$  of  $|J|$  such that when  $|J| < J_0$ , the temperature in SMA is always decreasing to its steady state, and when  $|J| > J_0$ , the temperature in SMA may not be always decreasing, which is an important property of thermoelectric SMA actuators reported by Lagoudas and Ding (1995). A lower bound of  $J_0$  is given. The physical implications of main results are also discussed.

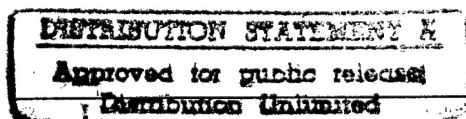
Key words: Heat transfer, thermoelectric, shape memory alloys, solution behavior, asymptotic behavior.

AMS(MOS) subject classifications: 35B05, 35B40, 80A15, 80A20.

Abbreviated titles: Solution behavior of the transient heat transfer problem in SMA.

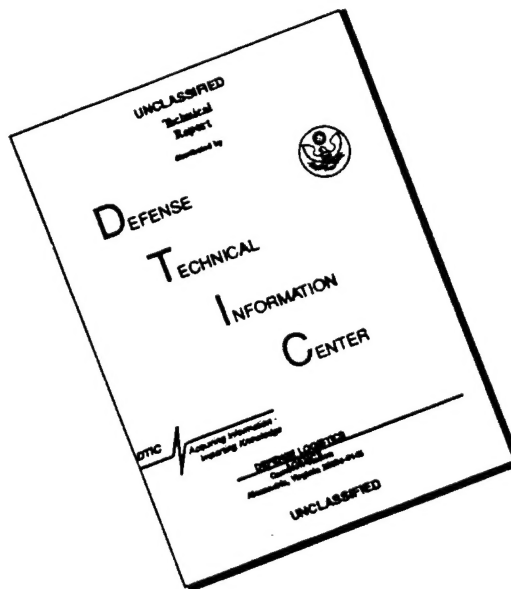
19960917 101

DTIC QUALITY INSPECTED 2



\* Department of Aerospace Engineering, Texas A&M University, College Station, TX 77843-3141.

# DISCLAIMER NOTICE



THIS DOCUMENT IS BEST QUALITY AVAILABLE. THE COPY FURNISHED TO DTIC CONTAINED A SIGNIFICANT NUMBER OF PAGES WHICH DO NOT REPRODUCE LEGIBLY.

# 1 INTRODUCTION

Shape Memory Alloys (SMA) [5] undergo an austenitic to martensitic phase transformation upon cooling, while with some hysteresis they fully recover their parent phase when heated above the austenitic finish temperature. A thermodynamic formulation of the phase transformation in SMA has been given by Tanaka and Nagaki [14] based on the thermodynamic approach to dissipation proposed by Edelen [9]. Many other researchers have also contributed to the thermomechanical constitutive description of SMA, some of the key references can be found in [2, 3]. SMA have been proposed as actuators for shape and vibration control of structures, especially the Ni-Ti (Nitinol) system. The capability of producing large actuation forces during the martensitic to austenitic phase transition is the major advantage of SMA. The major disadvantage in utilizing SMA actuators is the low rate of cooling, since the time constant for heat transfer is usually large compared to the frequencies required for many engineering applications. Recently a new SMA thermoelectrically cooled or heated actuator has been studied by Bhattacharyya et al. [1] that utilizes the thermoelectric effect for heat transfer from and to the SMA, by making the SMA actuator the cold or hot junction of a thermoelectric couple respectively. From the heat transfer model proposed in [1], Lagoudas and Ding [13] have proposed an equivalent simpler model, which is governed by an integro-differential equation, for 1-D symmetric thermoelectric SMA actuators. They investigated both transient and multiple cycle solutions induced by a piecewise constant electric current source numerically. They observed and conjectured that for the transient cooling problem with constant electric current density of magnitude  $|J|$  there is a value  $J_0$  of  $|J|$  such that when  $|J| > J_0$ , the temperature in SMA may not be always decreasing.

The purpose of this paper is to study the behavior of transient solutions to heat transfer problems of 1-D symmetric thermoelectric SMA actuators theoretically. We will restrict ourselves to the linear case (the heat capacity of SMA is approximated by a constant) and prove the above important conjecture of the transient solution and derive an estimation of the value  $J_0$ .

The arrangement of this paper is as follows. In Section 2, the analytical modeling is introduced and the existence and uniqueness of transient solutions are established. A numerical example corresponding to the typical Nitinol SMA actuator is also included to display the behavior of transient solutions. In Section 3, we first discuss the asymptotic behavior of the transient solutions and then prove that for the transient cooling problem with constant electric current density of magnitude  $|J|$  there is a value  $J_0$  of  $|J|$  such that when  $|J| > J_0$ , the temperature in SMA may not be always decreasing. An estimation of the value  $J_0$  is also given.

## 2 HEAT TRANSFER MODEL OF THERMOELECTRIC SMA ACTUATORS

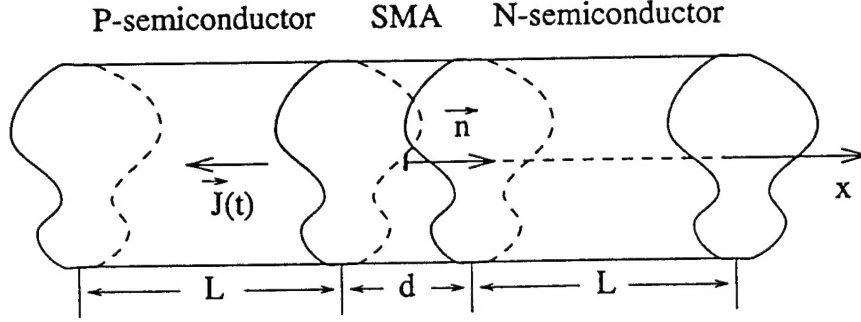


Figure 1: One dimensional three phase P-SMA-N thermoelectric system

The 1-D heat conduction problem in which a thin SMA material is confined between two semiconductors, a N-type and a P-type, forming the junction of a thermoelectric element is shown in Fig. 1. This thermoelectric SMA element is proposed by [1] as an efficient way of fast cooling or heating of SMA actuators. The principle of thermoelectric heat transfer is based upon the Peltier effect [8], according to which an electric current flow creates a temperature differential between junctions where dissimilar metals meet. For the case shown in Fig. 1, the net heat flux across interfaces, N-semiconductor/SMA and P-semiconductor/SMA, is given by

$$\begin{cases} (\vec{Q}_S - \vec{Q}_N) \cdot \vec{n} = (\alpha^N - \alpha^S)(\vec{J} \cdot \vec{n})T_N(\frac{d}{2}, t), \\ (\vec{Q}_S - \vec{Q}_P) \cdot \vec{n} = (\alpha^P - \alpha^S)(\vec{J} \cdot \vec{n})T_P(-\frac{d}{2}, t), \end{cases} \quad (2.1)$$

where, for  $i = N, S$  and  $P$  corresponding to N-semiconductor, SMA and P-semiconductor phases respectively,  $\vec{Q}_i$  is the heat flux vector in the  $i$ -th phase,  $\vec{n}$  is a unit normal along the positive  $x$ -direction (as shown in Fig. 1),  $\alpha^i$  is the Seebeck coefficient of  $i$ -th phase,  $T_i(x, t)$  is the temperature at  $(x, t)$  in  $i$ -th phases, and  $\vec{J}$  is the current density. When current  $\vec{J}$  flows through the junction, the Peltier effect induces jump discontinuities in the heat flux across the interfaces, which makes the thermoelectric element in Fig. 1 either a heating or cooling element, depending on the current direction.

In the 1-D three phase N-SMA-P thermoelectric system, it is assumed that the temperature in each phase does not significantly varies on the plane perpendicular to  $x$ -axis, hence

the Fourier law of heat conduction becomes

$$\vec{Q}_i = -K_i \frac{\partial T_i}{\partial x} \vec{n},$$

where  $K_i$  is the thermal conductivity of  $i$ -th phase. The electric current density vector is given by  $\vec{J}(x, t) = J(t)\vec{n}$ . If  $T_0$  is the surrounding temperature (for example, the room temperature) of the 1-D three phase N-SMA-P thermoelectric system, the convective heat transfer, which occurs across the sides of the  $i$ -th phase, is included approximately as a source term,  $-HP(T_i - T_0)/A$ , in the heat conduction equation for the N-SMA-P system, where  $H$  is the heat convection coefficient,  $P$  and  $A$  are the perimeter and area of the cross section respectively, which are assumed to be independent of  $x$  (see [6]). The heat conduction equations for the three phases of the N-SMA-P system are then given by (see [1])

$$K_N \frac{\partial^2 T_N}{\partial x^2}(x, t) + \rho_N J^2(t) - H \frac{P}{A}(T_N(x, t) - T_0) = C_v^N \frac{\partial T_N}{\partial t}(x, t), \quad (2.2)$$

$$d/2 < x < L + d/2, \quad t > 0,$$

$$K_s \frac{\partial^2 T_s}{\partial x^2}(x, t) + \rho_s J^2(t) - H \frac{P}{A}(T_s(x, t) - T_0) = C_v^s \frac{\partial T_s}{\partial t}(x, t), \quad (2.3)$$

$$-d/2 < x < d/2, \quad t > 0,$$

$$K_P \frac{\partial^2 T_P}{\partial x^2}(x, t) + \rho_P J^2(t) - H \frac{P}{A}(T_P(x, t) - T_0) = C_v^P \frac{\partial T_P}{\partial t}(x, t), \quad (2.4)$$

$$-L - d/2 < x < -d/2, \quad t > 0.$$

where  $\rho_i$  is the electrical resistivity and  $\rho_i J^2(t)$  represents the Joule heat in the  $i$ -th phase, and  $C_v^i$  is the heat capacity per unit volume of  $i$ -th phase. The interface conditions consist of continuity of the temperature field

$$T_s(\frac{d}{2}, t) = T_N(\frac{d}{2}, t), \quad T_s(-\frac{d}{2}, t) = T_P(-\frac{d}{2}, t), \quad (2.5)$$

while the heat flux interface conditions (2.1) given earlier reduce to

$$\begin{cases} -K_s \frac{\partial T_s}{\partial x}(\frac{d}{2}, t) = -K_N \frac{\partial T_N}{\partial x}(\frac{d}{2}, t) + \alpha_N T_N(\frac{d}{2}, t) J(t), \\ -K_s \frac{\partial T_s}{\partial x}(-\frac{d}{2}, t) = -K_P \frac{\partial T_P}{\partial x}(-\frac{d}{2}, t) + \alpha_P T_P(-\frac{d}{2}, t) J(t), \end{cases} \quad (2.6)$$

where it is assumed that  $\alpha_s = 0$ , which is justified as the Seebeck coefficient of an SMA is very low compared to that of the N or P semiconductor. Moreover, the end boundary conditions are

$$T_N(L + \frac{d}{2}, t) = T_0, \quad T_P(-L - \frac{d}{2}, t) = T_0, \quad (2.7)$$

while the initial conditions are stated as

$$\begin{cases} T_N(x, 0) = T_0, & \frac{d}{2} \leq x \leq L + \frac{d}{2}, \\ T_s(x, 0) = T_0, & -\frac{d}{2} \leq x \leq \frac{d}{2}, \\ T_P(x, 0) = T_0, & -L - \frac{d}{2} \leq x \leq -\frac{d}{2}. \end{cases} \quad (2.8)$$

The heat capacities,  $C_v^N$  and  $C_v^P$ , of N-type and P-type semiconductors are taken to be independent of temperature, while the heat capacity  $C_v^s$  of the SMA material varies significantly with  $T$  due to the temperature induced phase transformation. Even though the electrical resistivity  $\rho_s$  changes with temperature during the phase transformation of SMA, lack of precise experimental data limits us in assuming that  $\rho_s$  is constant. Dependence of  $\rho_s$  on temperature with possible hysteresis can be accounted for mathematically by varying the magnitude  $|J|$  of  $J$ .

Due to the relatively large thermal conductivity  $K_s$  of the SMA, compared with the thermal conductivities  $K_P$  of the P-semiconductor and  $K_N$  of the N-semiconductor, the temperature distribution in the SMA material is expected to be almost constant for small ratios  $d/L$ . It is possible, therefore, to simplify the three phase model into an approximate two phase model, by incorporating the SMA into the P-N interface. Integrating the field equation (2.3) with respect to  $x$  from  $-d/2$  to  $d/2$ , substituting (2.6) in the resulting equation, and then using (2.5) and the assumption that  $T_s(x, t)$  is independent of  $x$ , we obtain the following equation

$$\begin{aligned} & K_N \frac{\partial T_N}{\partial x}(\frac{d}{2}, t) - K_P \frac{\partial T_P}{\partial x}(-\frac{d}{2}, t) + \rho_s d J^2(t) - \frac{H d P}{A} (T_N(\frac{d}{2}, t) - T_0) \\ & = \alpha_N T_N(\frac{d}{2}, t) J(t) - \alpha_P T_P(-\frac{d}{2}, t) J(t) + C_v^s d \frac{dT_N}{dt}(\frac{d}{2}, t). \end{aligned} \quad (2.9)$$

Since the heat conduction equation (2.3) for SMA is replaced by the interface equation (2.9), for simplicity of notation, we shift the domains of the remaining field equations (2.2) and (2.4) for N-type and P-type semiconductors,  $[-L - \frac{d}{2}, -\frac{d}{2}]$  and  $[\frac{d}{2}, L + \frac{d}{2}]$ , to  $[-L, 0]$  and  $[0, L]$ ,

respectively. If we assume further that  $\alpha^P = -\alpha^N > 0$  and all other material parameters of  $P$ -type and  $N$ -type semiconductors are the same, namely,  $C_v^N = C_v^P = C_v$ ,  $K_N = K_P = K$  and  $\rho_N = \rho_P = \rho$ , then the temperature distributions  $T_N(x, t)$  and  $T_P(x, t)$  are symmetric, i.e.  $T_N(x, t) = T_P(-x, t)$ . Let  $T(x, t) = T_N(x, t) = T_P(-x, t)$ , the initial boundary value problem for the symmetric 1-D N-SMA-P system (2.2)-(2.8) is then reduced to

$$\left\{ \begin{array}{l} C_v \frac{\partial T}{\partial t}(x, t) = K \frac{\partial^2 T}{\partial x^2}(x, t) + \rho J^2(t) - \frac{HP}{A}(T(x, t) - T_0), \quad 0 < x < L, \quad t > 0 \\ \\ T(x, 0) = T_0, \quad 0 \leq x \leq L, \quad T(0, 0) = T_0, \\ \\ T(L, t) = T_0, \quad t \geq 0, \\ \\ 2K \frac{\partial T}{\partial x}(0, t) + \rho_s d J^2(t) - \frac{HdP}{A}(T(0, t) - T_0) \\ \\ = -2T(0, t)\alpha^P J(t) + C_v^s d \frac{dT}{dt}(0, t), \quad t > 0. \end{array} \right. \quad (2.10)$$

Because of the interface condition in equation (2.10), classical methods such as the separation of variables method, integral transform methods, etc. can not be directly applied to (2.10) to derive an analytical solution. Furthermore, because of the temperature dependence of the heat capacity of SMA, the boundary value problem (2.10) becomes extremely challenging to study theoretically and numerically.

One way to solve equation (2.10) is expressing the interface equation in (2.10) as an independent equation for  $T(0, t)$ , then solving the following Dirichlet boundary value problem (once the distribution  $T(0, t)$  is found):

$$\left\{ \begin{array}{l} C_v \frac{\partial T}{\partial t}(x, t) = K \frac{\partial^2 T}{\partial x^2}(x, t) + \rho J^2(t) - \frac{HP}{A}(T(x, t) - T_0), \quad 0 < x < L, \quad t > 0 \\ \\ T(x, 0) = T_0, \quad 0 \leq x \leq L, \\ \\ T(L, t) = T_0, \quad t \geq 0, \\ \\ T(0, t) = T(t), \quad t \geq 0. \end{array} \right. \quad (2.11)$$

To express the interface equation in (2.10) as an independent equation for  $T(0, t)$ , we need to find the relation between  $T(0, t)$  and  $\frac{\partial T}{\partial x}(0, t)$ . Assume  $T(0, t) = T(t)$  is given in (2.11),

by using the separation of variables method and Fourier series technique, we could obtain  $T(x, t) = H(x, t, T(t))$  from (2.11). Hence the relation between  $T(0, t)$  and  $\frac{\partial T}{\partial x}(0, t)$  would be given by  $\frac{\partial T}{\partial x}(0, t) = \frac{\partial H}{\partial x}(0, t, T(t))$ . Based on this idea, we obtain the following integro-differential equation of  $T(0, t) = T(t)$ , which is equivalent to the interface equation in (2.10) (the detailed calculation can be found in [13]),

$$\begin{cases} \int_0^t G(t-\tau) \left( \frac{dT}{d\tau}(\tau) + \nu_1 T(\tau) \right) d\tau + \mu_1 \frac{dT}{dt}(t) + \nu_2(t) T(t) = T_0 \mathcal{F}(t) \\ T(0) = T_0, \end{cases} \quad (2.12)$$

where

$$\mu_1 = \frac{C_v^s dL}{4K}; \quad \nu_1 = \frac{HP}{C_v A}; \quad \nu_2(t) = \frac{-L\alpha^P J(t)}{2K} + \frac{1}{2} + \frac{PHLd}{4KA};$$

$$F(t) = \sum_{n=1}^{\infty} \text{EXP} \left\{ - \left( \frac{K}{C_v} \left( \frac{(2n-1)\pi}{L} \right)^2 + \frac{HP}{C_v A} \right) t \right\};$$

$$G(t) = \sum_{n=1}^{\infty} \text{EXP} \left\{ - \left( \frac{K}{C_v} \left( \frac{n\pi}{L} \right)^2 + \frac{HP}{C_v A} \right) t \right\};$$

and

$$\mathcal{F}(t) = \int_0^t G(t-\tau) \frac{HP}{C_v A} d\tau + \frac{2\rho}{T_0 C_v} \int_0^t F(t-\tau) J^2(\tau) d\tau + \frac{1}{2} + \frac{PHLd}{4KA} + \frac{\rho_s dL}{4KT_0} J^2(t).$$

Once the temperature distribution in the SMA interface  $T(0, t) = T(t)$  is found from (2.12), the temperature distribution  $T(x, t)$  in  $P$ -semiconductor and  $N$ -semiconductor can be found by solving the initial boundary value problem (2.11). Therefore the study of the transient heat conduction for the 1-D thermoelectric P-SMA-N system reduces to the analysis of the integro-differential equation (2.12). Before we proceed with further discussions, it is desirable, for simplicity of mathematical and physical discussion, to non-dimensionalize equation (2.12) by using non-dimensional groups which are obtained by the application of the Buckingham- $\pi$  theorem [4]. The non-dimensional quantities in this study following the notation used in [1], where a letter with an overbar represents the non-dimensional

counterpart of a physical quantity with same letter, are listed below:

$$\left\{ \begin{array}{lll} \bar{x} = \frac{x}{L}; & \bar{t} = t \left( \frac{C_v L^2}{K} \right)^{-1}; & \bar{T} = \frac{T}{T_0}; \\ \bar{J} = JL \sqrt{\frac{\rho}{KT_0}}; & \bar{\alpha} = \alpha^P \sqrt{\frac{T_0}{K\rho}}; & \bar{H} = \frac{PHL^2}{KA}; \\ \bar{\rho} = \frac{\rho_s}{\rho}; & \bar{C}_v^s = \frac{C_v^s}{C_v}; & \bar{d} = \frac{d}{2L}. \end{array} \right. \quad (2.13)$$

From now on, unless mentioned otherwise, all quantities are assumed to be non-dimensional. For simplicity of notation, we will drop the overbar from the letters used for the non-dimensionalized quantities introduced in (2.13). Then equation (2.12) becomes

$$\left\{ \begin{array}{l} \int_0^t G(t-\tau) \left( \frac{dT}{d\tau}(\tau) + HT(\tau) \right) d\tau + \mu \frac{dT}{dt}(t) + \nu(t)T(t) = \mathcal{F}(t) \\ T(0) = 1, \end{array} \right. \quad (2.14)$$

where

$$\mu = \frac{C_v^s d}{2}; \quad \nu(t) = \frac{1}{2} + \frac{1}{2}Hd - \frac{1}{2}\alpha J(t); \quad (2.15)$$

$$F(t) = \sum_{n=1}^{\infty} e^{-((2n-1)^2\pi^2 + H)t}; \quad G(t) = \sum_{n=1}^{\infty} e^{-(n^2\pi^2 + H)t};$$

and

$$\mathcal{F}(t) = H \int_0^t G(\tau) d\tau + 2 \int_0^t F(t-\tau) J^2(\tau) d\tau + \frac{1}{2} + \frac{1}{2}Hd + \frac{1}{2}\rho d J^2(t).$$

The existence and uniqueness of solution to (2.14) have to be established first.

**Theorem 2.1** *Let  $J \in C(0, \infty)$  and the heat capacity of SMA,  $C_v^s$ , be a constant. Then the integro-differential equation (2.14) admits a unique solution in  $C[0, \infty)$ .*

**Proof.** Let us first consider the following equation

$$\left\{ \begin{array}{l} \frac{dT}{dt}(t) + HT(t) = S(t), \quad t > 0 \\ T(0) = 1. \end{array} \right. \quad (2.16)$$

We obtain

$$T(t) = e^{-Ht} + \int_0^t e^{-H(t-\tau)} S(\tau) d\tau. \quad (2.17)$$

Substitute  $T(t)$  into equation (2.14), we have the following integral equation,

$$\mu S(t) + \int_0^t G^*(t, \tau) S(\tau) d\tau = \mathcal{F}^*(t), \quad (2.18)$$

where

$$G^*(t, \tau) = G(t - \tau) + (\nu(t) - \mu H) e^{-H(t - \tau)},$$

$$\mathcal{F}^*(t) = \mathcal{F}(t) - (\nu(t) - \mu H) e^{-Ht}.$$

Equation (2.18) is a Volterra equation of the second kind, where the kernel  $G^*(t, \tau)$  has a weak singularity of  $|t - \tau|^{-\frac{1}{2}}$ . By [11, Theorem 7, pp. 35-37], equation (2.18) has a unique solution in  $C(0, \infty)$ . I

Table 1: Material parameters

	Ni-Ti SMA	P-Element
Thermal Conductivity $K$	$2.2 \times 10^{-2} W / (mm - ^\circ K)$	$1.63 \times 10^{-3} W / (mm - ^\circ K)$
Heat Capacity $C_v$	$2.12 \times 10^{-3} J / (mm^3 - ^\circ K)$	$4.35 \times 10^{-3} J / (mm^3 - ^\circ K)$
Resistance Density $\rho$	$6.3242 \times 10^{-4} \Omega - mm$	$1.15 \times 10^{-2} \Omega - mm$
Seebeck Coefficient $\alpha$	$1.2 \times 10^{-5} Volts / ^\circ K$	$2.15 \times 10^{-4} Volts / ^\circ K$

Let us also look at an example of a typical Ni-Ti SMA actuator to get some intuition. The material parameters of the Ni-Ti SMA and the P-type semiconductor at room temperature  $T_0 = 300^\circ K$  shown in Table 1 are adopted from [1]. The heat convection coefficient  $H$  is assumed to be  $H = 2.5 \times 10^{-5} W / (mm^2 - ^\circ K)$ , and the geometric parameters of the SMA-actuator are given by  $L = 4.0 mm$ ,  $d = 2.0 mm$ ,  $P = 4.0 mm$  and  $A = 1.0 mm^2$ . By (2.13), we obtain the following non-dimensionalized parameters:

$$\begin{aligned} \rho &= 5.4993 \times 10^{-2}, & \alpha &= 0.8601, & C_v^s &= 0.4874, \\ H &= 0.9816, & d &= 0.25. \end{aligned}$$

Applying the finite difference scheme to (2.14) [13], we obtain the temperature distribution at the SMA interface corresponding to different values of  $J$  for constant  $C_v^s$ , which is shown in Figure 2.

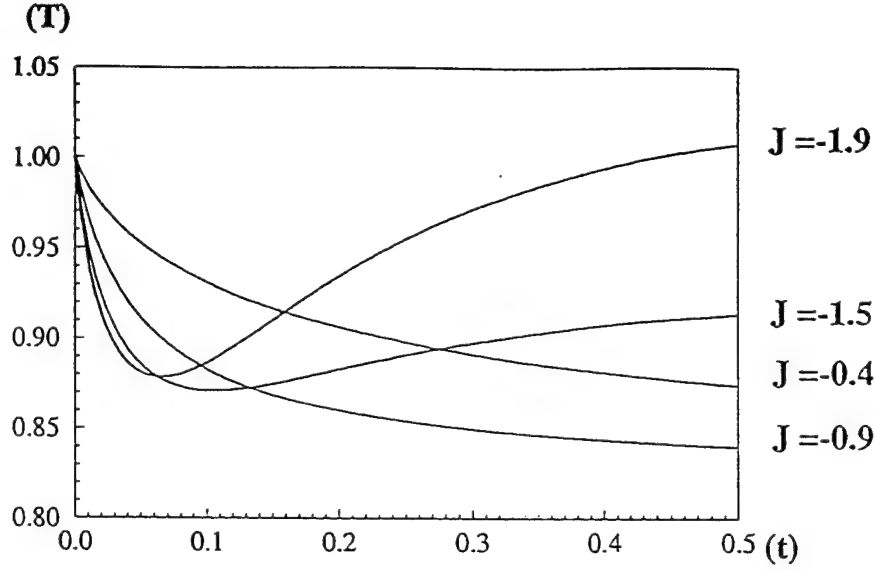


Figure 2: Temperature in SMA corresponding to different values of  $J$  for constant  $C_v^s$

The temperature distributions at the SMA interface corresponding to different values of  $J$  for variable  $C_v^s$  is shown in Fig. 3, where the dimensional heat capacity of a typical Ni-Ti SMA has been experimentally determined and may be represented by the function (see [1, 12])

$$C_v^s = C_v^0 + [L_H - C_v^0(M_s - M_f)] \cdot \frac{\ln 100}{|M_s - M_f|} e^{-\frac{2 \ln 100}{|M_s - M_f|} |T - \frac{M_s + M_f}{2}|} \quad M_f \leq T \leq M_s, \quad (2.19)$$

during the forward transformation (austenite to martensite) and the function

$$C_v^s = C_v^0 + [L_H - C_v^0(A_f - A_s)] \cdot \frac{\ln 100}{|A_s - A_f|} e^{-\frac{2 \ln 100}{|A_s - A_f|} |T - \frac{A_s + A_f}{2}|} \quad A_f \leq T \leq A_s, \quad (2.20)$$

during the reverse transformation (martensite to austenite), with  $C_v^s = C_v^0$  at all other temperatures. When an SMA is initially at the austenitic state, its crystal structure, upon cooling, starts to change from the austenitic phase to the martensitic phase at the starting temperature,  $M_s$ , and finishes at the finishing temperature,  $M_f$ , when the martensitic phase is formed. When an SMA is initially at the martensitic state, its crystal structure, upon heating, starts to change from the martensitic phase to the austenitic phase at the starting temperature,  $A_s$ , and finishes at the finishing temperature,  $A_f$ , when the austenitic phase is formed. For a typical Ni-Ti SMA [1],  $M_f = 278^\circ K$ ,  $M_s = 296^\circ K$ ,  $A_s = 302^\circ K$  and  $A_f = 324^\circ K$ ,  $L_H = 0.1 \text{ J/mm}^3$  and  $C_v^0 = 2.12 \times 10^{-3} \text{ J/(mm}^3 \cdot ^\circ K)$ .

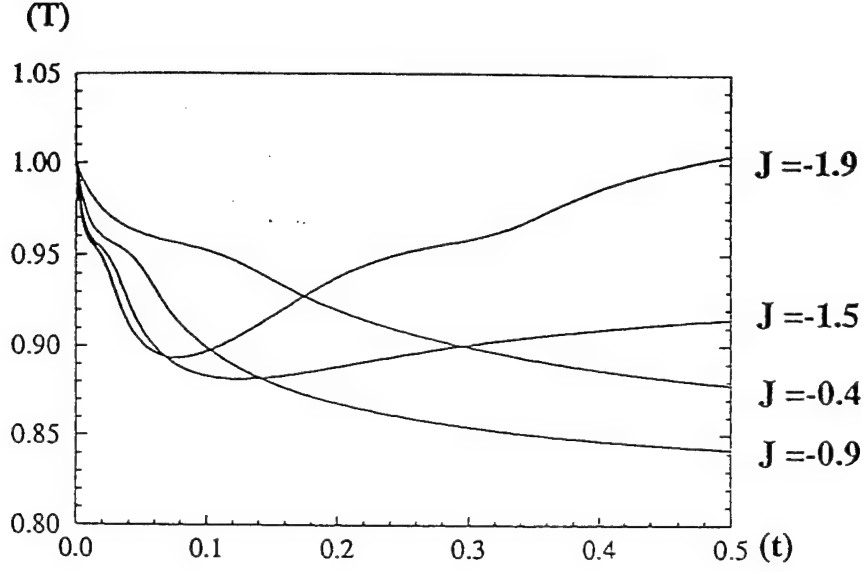


Figure 3: Temperature in SMA corresponding to different current densities for variable  $C_v^s$

For the constant  $C_v^s$  case, we find from Fig. 2 that the temperature profiles in SMA are quite different corresponding to different values of the current density  $J$ . For  $J = -0.4$  and  $J = -0.9$ , the corresponding temperature in SMA always decreases asymptotically to the steady state temperature. For  $J = -1.5$  and  $J = -1.9$ , the corresponding temperature in SMA decreases first, it reaches its lowest value at some finite time and then asymptotically increases to the steady state temperature. Similarly, for the variable  $C_v^s$  case, we observe the same phenomena as shown in Fig. 3 and discussed in more detail in [13]. As pointed out before, the major disadvantage in utilizing SMA actuators is the low rate of cooling, because the time constant for heat transfer is usually large compared to the small time constants required for many high frequency engineering applications. Thus to maintain a monotonic decay of the SMA temperature is very important in engineering applications. It is conjectured and confirmed numerically in [13] that for the transient cooling problem with constant electric current density of magnitude  $|J|$  there is a value  $J_0$  of  $|J|$  such that when  $|J| < J_0$ , the temperature in SMA is always decreasing to its steady state, and when  $|J| > J_0$ , the temperature in SMA may not be always decreasing. We will further investigate this question in the next section.

### 3 SOLUTION BEHAVIOR OF THE TRANSIENT HEAT TRANSFER PROBLEM

In this section, we investigate the solution behavior of (2.14). Due to the complicated nature of SMA heat capacity  $C_v^s$  (see (2.19) and (2.20)), we only study the constant  $C_v^s$  case. Thus through this section, we always assume  $C_v^s$  is constant. In most of engineering applications of thermoelectric SMA actuators, the typical time for current density  $J(t)$  to reach its steady state is much smaller than the typical time for temperature  $T(t)$  to reach its steady state. Therefore for the study of long-time asymptotic behavior of  $T(t)$ , we will assume  $J(t)$  to be constant. For the study of small-time asymptotic behavior of  $T(t)$ , we will not assume  $J(t)$  to be constant. If  $J(t)$  is assumed to be constant, the exact solution of (2.14) can be found by using the Laplace transform. However, the expression is too complicated for an immediate interpretation.

Let us first look at the long-time asymptotic behavior of the transient solution to (2.14).

**Theorem 3.1** *Let  $J(t) = J$  be a constant function in equation (2.14) and  $T(t)$  be the solution of (2.14). Then the transient solution  $T(t)$  is stable provided that*

$$J < J_{max} \equiv \frac{2H\eta_1 + 1 + Hd}{\alpha}.$$

*The steady state solution is given by the following formula,*

$$T_\infty = \frac{(2H\eta_1 + 1 + Hd) + (\rho d + 4\eta_2) J^2}{(2H\eta_1 + 1 + Hd) - \alpha J},$$

*where  $\eta_1$  and  $\eta_2$  are given by*

$$\eta_1 = \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2 + H}, \quad \eta_2 = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2\pi^2 + H}. \quad (3.1)$$

**Proof.** To investigate the stability of the solution to equation (2.14), we only need to study the transfer function of (2.14),  $\hat{w}(s)$ , which is the Laplace transform of  $w(t)$ , the solution of (2.14) corresponding to the unit-impulse input at  $t = 0$ . Applying the Laplace transform to the equation

$$\int_0^t G(t-\tau) \left( \frac{dw}{d\tau}(\tau) + Hw(\tau) \right) d\tau + \mu \frac{dw}{dt}(t) + \nu w(t) = \delta(t),$$

where  $\delta(t)$  is the unit-impulse function at  $t = 0$ , we obtain the transfer function  $\hat{w}(s)$  of (2.14) given by

$$\hat{w}(s) = \frac{1}{\hat{G}(s)(s+H) + \mu s + \nu},$$

where

$$\hat{G}(s) = \sum_{n=1}^{\infty} \frac{1}{s + n^2\pi^2 + H}.$$

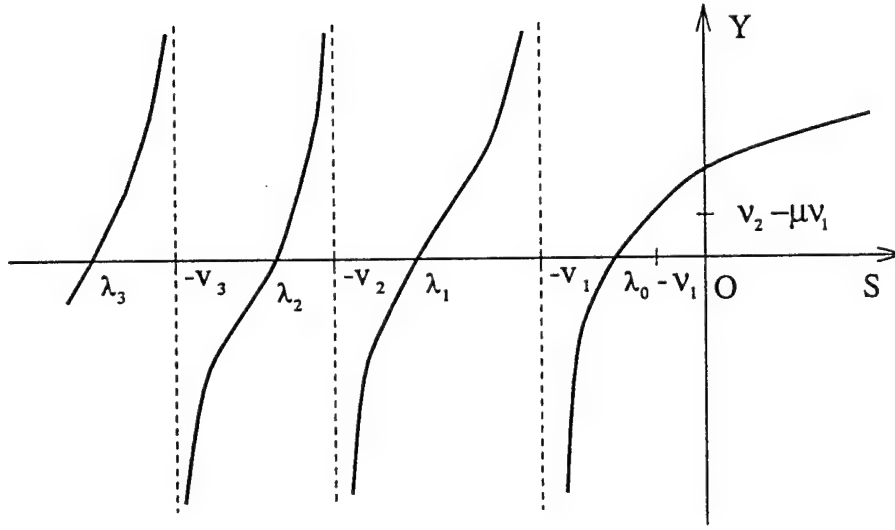


Figure 4: Graph of  $\hat{G}(s)(s+H) + \mu s + \nu$

For simplicity of notations, let

$$v_n = n^2\pi^2 + H, \quad n \geq 1.$$

Notice that the denominator of  $\hat{w}(s)$ ,  $\hat{G}(s)(s+H) + \mu s + \nu$ , is continuous on  $(-v_{n+1}, -v_n)$ ,  $n = 1, 2, \dots$ , and  $(-v_1, +\infty)$ ;  $\hat{G}(s)(s+H) + \mu s + \nu \rightarrow \mp\infty$ , when  $s \rightarrow v_n \pm 0$  for any  $n \geq 1$ ; and

$$(\hat{G}(s)(s+H) + \mu s + \nu)' = \sum_{n=1}^{\infty} \frac{v_n - H}{(s + v_n)^2} + \mu > 0.$$

Thus  $\hat{G}(s)(s+H) + \mu s + \nu$  is always increasing. The graph of  $\hat{G}(s)(s+H) + \mu s + \nu$  is sketched in Fig. 4. Denote by  $\{\lambda_n\}_{n=0}^{\infty}$  the zeros of  $\hat{G}(s)(s+H) + \mu s + \nu$  on the real line  $\Re$ . Thus

$$\dots < \lambda_n < -v_n < \lambda_{n-1} < -v_{n-1} < \dots < -v_2 < \lambda_1 < -v_1 < 0,$$

and  $\lambda_n \simeq O(n^2)$ . It is also easy to check that  $\hat{G}(s)(s + H) + \mu s + \nu$  has no other zero on the complex plane. Thus except for  $\{\lambda_n\}_{n=0}^\infty$ ,  $\hat{w}(s)$  has no other poles on the complex plane. For the first pole  $\lambda_0$  of  $\hat{w}(s)$ ,  $\lambda_0 < 0$  if and only if

$$\hat{G}(0)(0 + H) + \mu \cdot 0 + \nu = \hat{G}(0)H + \nu > 0.$$

By using notations in (2.15) and (3.1), we have

$$H\eta_1 + \frac{1}{2} + \frac{1}{2}Hd - \frac{1}{2}\alpha J > 0.$$

Thus the transient solution  $T(t)$  of (2.14) is stable if and only if

$$J < \frac{2H\eta_1 + 1 + Hd}{\alpha}.$$

When  $J$  satisfies the above inequality, the steady state of the transient solution  $T(t)$  to equation (2.14) can be obtained. It is easy to verify that

$$\lim_{t \rightarrow \infty} \mathcal{F}(t) = \left(H\eta_1 + \frac{1}{2} + \frac{1}{2}Hd\right) + \left(\frac{1}{2}\rho d + 2\eta_2\right) J^2.$$

Thus letting  $\frac{dT}{dt}(t) = 0$  and  $t \rightarrow \infty$  in (2.14), we obtain that

$$T_\infty = \frac{(2H\eta_1 + 1 + Hd) + (\rho d + 4\eta_2) J^2}{(2H\eta_1 + 1 + Hd) - \alpha J}.$$

■

**Remark 3.1** Note that in the present work, the heat convection coefficient  $H$  is assumed to be constant, which is true for small temperature variations. For the example given in the last section, for a stable transient solution to exist,  $J$  has to be less than  $J_{max} = 1.8056$ . Since  $T_\infty$  approaches infinity when  $J$  is close to  $J_{max}$  or goes to  $-\infty$ , as shown in Fig. 5, the assumption of  $H$  being constant is not valid any longer.

**Remark 3.2** Theorem 3.1 gives an important condition on the current density  $J$  for the stable temperature of (2.14). For the transient cooling problem of SMA actuator, i.e.  $J \leq 0$ , the transient solution is always stable. Corresponding to the example given in the last section with the data given in Table 1, we have

$$T_\infty(J) = \frac{1.5530 + 0.4765J^2}{1.5530 - 0.8601J}.$$

From the graph of  $T_\infty(J)$ , shown in Fig. 5, we have that  $T_\infty^{min} \equiv \min_{J < 0} T_\infty(J) = 0.8285$  when  $J_\infty^{min} = -0.7476$ .

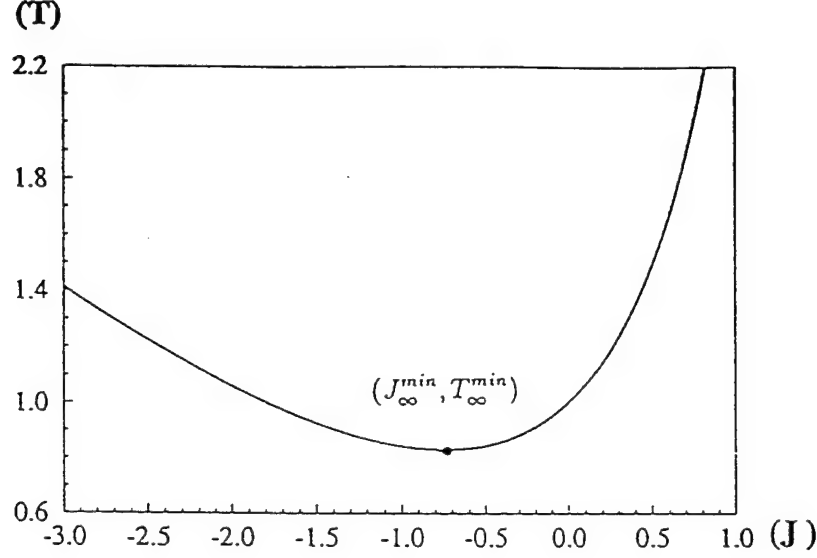


Figure 5: The graph of  $T_\infty$

Even though  $T_\infty^{\min}$  is a measure of the maximum ability of the thermoelectric element to cool down the N-SMA-P junction, the short-time response is also important for the design of high frequency SMA actuators, and it will be investigated next. Let  $t_0 > 0$  be very small, and let  $J(t)$  be given by

$$J(t) = J_1 + J_2 t + o(t), \quad 0 \leq t < t_0.$$

Assume that  $T(t)$ , for small time  $t$ , has the asymptotic series expansion given by

$$T(t) = 1 + T_1 t + o(t), \quad 0 \leq t < t_0.$$

We will determine  $T_1$  from (2.14) in terms of  $J_1$ ,  $J_2$  and material parameters. Substituting the series expansions of  $T(t)$  and  $J(t)$  into (2.14), we have

$$\begin{aligned} & \int_0^t G(t - \tau) (T_1 + H + o(1)) d\tau + \mu T_1 + o(1) \\ & + \left( -\frac{1}{2} \alpha (J_1 + J_2 t + o(t)) + \frac{1}{2} + \frac{1}{2} H d \right) (1 + T_1 t + o(t)) = \mathcal{F}(t). \end{aligned}$$

By comparing the coefficients of 0-th order terms in  $t$  and using the fact that

$$0 \leq F(t) \leq G(t) \leq \frac{1}{\pi \sqrt{t}}, \quad t > 0, \quad (3.2)$$

we obtain

$$T_1 = \left( \frac{\alpha}{2\mu} + \frac{\rho d}{2\mu} J_1 \right) J_1 \approx \frac{\alpha J_1}{2\mu} = \frac{\alpha J_1}{C_v^s d},$$

where the approximate value is justified whenever  $\frac{\rho d J_1}{\alpha} \ll 1$ , which is the case for the example given earlier.

If  $J_1 = 0$ , hence  $T_1 = 0$ , the above expansion of  $T(t)$  does not provide any further information. Let  $J(t)$  be given by

$$J(t) = J_2 t + o(t), \quad 0 \leq t < t_0.$$

Assume that  $T(t)$ , for small time  $t$ , has the asymptotic series expansion given by

$$T(t) = 1 + T_2 t^2 + o(t^2), \quad 0 \leq t < t_0.$$

Substituting the series expansions of  $T(t)$  and  $J(t)$  into (2.14), we have

$$\begin{aligned} & \int_0^t G(t-\tau) (H + 2T_2 t + o(t)) d\tau + 2\mu T_2 t + o(t) \\ & + \left( -\frac{1}{2}\alpha(J_2 t + o(t)) + \frac{1}{2} + \frac{1}{2}Hd \right) (1 + T_2 t^2 + o(t^2)) = \mathcal{F}(t). \end{aligned}$$

By comparing the coefficients of first order terms in  $t$  and using (3.2), we obtain

$$T_2 = \frac{\alpha}{4\mu} J_2 = \frac{\alpha}{2C_v^s d} J_2.$$

Next we will study the solution behavior of equation (2.14). The following important theorem explains why the solution behavior of equation (2.14) is quite different for different values of  $J$ .

**Theorem 3.2** *Let  $T(t)$  be the solution of equation (2.14) with  $J(t) = J \leq 0$ , corresponding to the transient cooling problem. Then there exists a value  $J_0$  for  $|J|$ , such that when  $0 \leq |J| \leq J_0$ ,  $T(t)$  is always decreasing to its steady state  $T_\infty$ , otherwise  $T(t)$  may not be always decreasing. A lower bound of  $J_0$  is given by*

$$J_0^* = \frac{-b + \sqrt{b^2 + 4ac}}{2a},$$

where

$$a = \frac{2\sqrt{3}\alpha}{\pi^2},$$

$$b = \rho d + \frac{\sqrt{3}}{\pi} + \frac{2\sqrt{3}}{\pi^2} [1 + Hd(1 - C_v^*)],$$

$$c = \alpha.$$

**Proof.** Let us consider the homogeneous equation

$$\begin{cases} \int_0^t G(t-\tau) \left( \frac{dT_1}{d\tau}(\tau) + HT_1(\tau) \right) d\tau + \mu \frac{dT_1}{dt}(t) + \nu T_1(t) = 0 \\ T_1(0) = 1. \end{cases} \quad (3.3)$$

Applying the Laplace transform to this equation, we have

$$\hat{T}_1(s) = \frac{\mu + \hat{G}(s)}{\hat{G}(s)(s + H) + \mu s + \nu}.$$

By Lemma A.2 in the Appendix, there exists a sequence of real numbers  $\{b_n\}$  such that

$$\hat{T}_1(s) = \sum_{n=0}^{\infty} \frac{b_n}{s - \lambda_n}.$$

Then

$$T_1(t) = \sum_{n=0}^{\infty} b_n e^{\lambda_n t}, \quad (3.4)$$

and  $T_1(0) = 1$ .

Now consider the non homogeneous equation

$$\begin{cases} \int_0^t G(t-\tau) \left( \frac{dT_2}{d\tau}(\tau) + HT_2(\tau) \right) d\tau + \mu \frac{dT_2}{dt}(t) + \nu T_2(t) = \mathcal{F}(t) \\ T_2(0) = 0. \end{cases} \quad (3.5)$$

By using the transfer function and Lemma A.1 in the Appendix, we have

$$T_2(t) = \sum_{n=0}^{\infty} a_n \int_0^t e^{\lambda_n(t-\tau)} \mathcal{F}(\tau) d\tau. \quad (3.6)$$

Thus the solution of equation (2.14) is given by

$$T(t) = T_1(t) + T_2(t) = \sum_{n=0}^{\infty} b_n e^{\lambda_n t} + \sum_{n=0}^{\infty} a_n \int_0^t e^{\lambda_n(t-\tau)} \mathcal{F}(\tau) d\tau. \quad (3.7)$$

Rewrite  $\mathcal{F}(t)$  as

$$\mathcal{F}(t) = \mathcal{F}(\infty) - H \sum_{n=1}^{\infty} \frac{1}{v_n} e^{-v_n t} - 2J^2 \sum_{n=1}^{\infty} \frac{1}{v_{2n-1}} e^{-v_{2n-1} t},$$

where

$$\mathcal{F}(\infty) = H\eta_1 + 2J^2\eta_2 + \frac{1}{2} + \frac{1}{2}Hd + \frac{1}{2}\rho dJ^2. \quad (3.8)$$

By (3.7), we have

$$T'(t) = \sum_{n=0}^{\infty} \lambda_n b_n e^{\lambda_n t} + \sum_{n=0}^{\infty} a_n \lambda_n \int_0^t e^{\lambda_n(t-\tau)} \mathcal{F}(\tau) d\tau + \mathcal{F}(t) \sum_{n=0}^{\infty} a_n.$$

Substituting the expression of  $\mathcal{F}(t)$  in  $T'(t)$ , we obtain

$$T'(t) = \sum_{n=0}^{\infty} C_n^1 e^{\lambda_n t} + \sum_{m=1}^{\infty} C_m^2 e^{-v_m t} + \sum_{m=1}^{\infty} C_m^3 e^{-v_{2m-1} t}, \quad (3.9)$$

where

$$\begin{aligned} C_n^1 &= \lambda_n b_n + \mathcal{F}(\infty) a_n - H a_n \lambda_n \sum_{m=1}^{\infty} \frac{1}{v_m (\lambda_n + v_m)} \\ &\quad - 2J^2 a_n \lambda_n \sum_{m=1}^{\infty} \frac{1}{v_{2m-1} (\lambda_n + v_{2m-1})}; \\ C_m^2 &= H \sum_{n=0}^{\infty} \frac{a_n \lambda_n}{v_m (\lambda_n + v_m)} - \frac{H}{v_m} \sum_{n=0}^{\infty} a_n; \\ C_m^3 &= 2J^2 \sum_{n=0}^{\infty} \frac{a_n \lambda_n}{v_{2m-1} (\lambda_n + v_{2m-1})} - \frac{2J^2}{v_{2m-1}} \sum_{n=0}^{\infty} a_n. \end{aligned}$$

Let us first consider  $\{C_m^2\}$ . Simplifying  $C_m^2$ , we obtain

$$C_m^2 = -H \sum_{n=0}^{\infty} \frac{a_n}{\lambda_n + v_m}.$$

Let  $s = -v_m$  in (A.1), we have

$$\sum_{n=0}^{\infty} \frac{a_n}{-v_m - \lambda_n} = 0.$$

Hence  $C_m^2 = 0$ . By the same reason, we also have  $C_m^3 = 0$ . Let us estimate  $C_n^1$ . Simplifying the expression of  $C_n^1$  and substituting (3.8), we obtain

$$C_n^1 = \lambda_n b_n + \nu a_n + \frac{1}{2}(\rho d J^2 + \alpha J) a_n + H a_n \hat{G}(\lambda_n) + 2J^2 a_n \sum_{m=1}^{\infty} \frac{1}{\lambda_n + v_{2m-1}}. \quad (3.10)$$

By Lemma A.2 in the Appendix, substituting  $b_n = (\hat{G}(\lambda_n) + \mu) a_n$  into (3.10) and using the fact that  $\hat{G}(\lambda_m)(\lambda_m + H) + \mu \lambda_m + \nu = 0$  for any  $m \geq 0$ , we obtain

$$C_n^1 = \frac{1}{2}(\rho d J^2 + \alpha J) a_n + 2J^2 a_n \sum_{m=1}^{\infty} \frac{1}{\lambda_n + v_{2m-1}}. \quad (3.11)$$

Thus

$$T'(t) = \sum_{n=0}^{\infty} \left( \frac{1}{2}(\rho d J^2 + \alpha J) a_n + 2J^2 a_n \sum_{m=1}^{\infty} \frac{1}{\lambda_n + v_{2m-1}} \right) e^{\lambda_n t}. \quad (3.12)$$

Let  $J = -|J|$ , where  $|J|$  is the magnitude of  $J$ . Suppose there is  $t_0 > 0$  such that  $T'(t_0) = 0$ , then it is easy to check that  $T''(t_0) > 0$ . Hence  $T(t)$  may have at most one local minimum point and no maximum point. By solving  $T'(t_0) = 0$  for  $|J|$ , we have

$$|J| = \frac{\alpha \sum_{n=0}^{\infty} a_n e^{\lambda_n t_0}}{\rho d \sum_{n=0}^{\infty} a_n e^{\lambda_n t_0} + 4 \sum_{n=0}^{\infty} a_n \left( \sum_{m=1}^{\infty} \frac{1}{\lambda_n + v_{2m-1}} \right) e^{\lambda_n t_0}}. \quad (3.13)$$

When  $t_0$  increases,  $|J|$  decreases. Note that

$$\dots < \lambda_n < \lambda_{n-1} < \dots < \lambda_1 < \lambda_0 < -H, \quad \text{and} \quad \lambda_n \simeq O(n^2).$$

Then by letting  $t_0 \rightarrow \infty$ , we have

$$|J| = \frac{\alpha}{\rho d + 4 \sum_{m=1}^{\infty} \frac{1}{\lambda_0 + v_{2m-1}}}. \quad (3.14)$$

Equation (3.14) is a nonlinear algebraic equation because  $\lambda_0$  is dependent of  $J$ . The existence of solution to (3.14) is easily obtained by using the Mean Value Theorem, while the uniqueness follows from the monotonicity of  $\lambda_0(J)$  and Rolle's Theorem. Denote the solution to (3.14) by  $J_0$ . When  $J > J_0$ , equation (3.13) will admit a solution for  $t_0$ , hence  $T'(t_0) = 0$ . Thus the solution,  $T(t)$ , to (2.14) with  $J > J_0$  decreases first, reaches its minimum value at

$t = t_0$  and then increases to its steady state  $T_\infty$ . On the other hand, the solution  $T(t)$ , to (2.14) with  $J < J_0$  decreases monotonically to its steady state  $T_\infty$  because  $T'(t) \neq 0$  for any finite value of  $t > 0$ .

Next we will give a lower bound estimate of the value  $J_0$ . Notice that

$$\sum_{m=1}^{\infty} \frac{1}{\lambda_n + v_{2m-1}} \leq \sum_{m=1}^{\infty} \frac{1}{\lambda_0 + v_{2m-1}}, \quad \forall n \geq 0.$$

Thus from (3.11), we have

$$C_n^1 < \frac{1}{2}(\rho d|J|^2 - \alpha|J|)a_n + 2|J|^2 a_n \sum_{m=1}^{\infty} \frac{1}{\lambda_0 + v_{2m-1}}. \quad (3.15)$$

We need to estimate the value of  $\lambda_0$ . Let  $s = -H - \theta(v_1 - H)$  where  $0 < \theta < 1$  to be determined. Then

$$\begin{aligned} & \hat{G}(s)(s + H) + \mu s + \nu \\ &= -\theta \sum_{n=2}^{\infty} \frac{1}{n^2 - \theta} - \frac{\theta}{1 - \theta} - \theta\mu(v_1 - H) + \nu - \mu H \\ &\leq -\frac{\theta}{1 - \theta} + \nu - \mu H. \end{aligned}$$

By solving the following inequality

$$-\frac{\theta}{1 - \theta} + \nu - \mu H \leq 0,$$

we obtain that

$$\theta \geq \frac{\nu - \mu H}{1 + \nu - \mu H}.$$

Thus by letting

$$\theta = \frac{\nu - \mu H}{1 + \nu - \mu H},$$

we have

$$-v_1 < -H - \frac{\nu - \mu H}{1 + \nu - \mu H}(v_1 - H) < \lambda_0 < -H. \quad (3.16)$$

Since  $\sum_{m=1}^{\infty} \frac{1}{s + v_{2m-1}}$  is a continuous and decreasing function on  $(-v_1, 0)$ , it follows from (3.15) that

$$\begin{aligned}
C_n^1 &\leq a_n \left( \frac{\rho d}{2} |J|^2 - \frac{\alpha}{2} |J| + 2|J|^2 \sum_{m=1}^{\infty} \frac{1}{-H - \frac{\nu - \mu H}{1 + \nu - \mu H} (v_1 - H) + v_{2m-1}} \right) \\
&= a_n \left( \frac{\rho d}{2} |J|^2 - \frac{\alpha}{2} |J| + \frac{2|J|^2}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{-\frac{\nu - \mu H}{1 + \nu - \mu H} + (2m-1)^2} \right) \\
&= a_n \left( \frac{\rho d}{2} |J|^2 - \frac{\alpha}{2} |J| + \frac{2|J|^2}{\pi^2} \left( \frac{\pi}{4\sqrt{\frac{\nu - \mu H}{1 + \nu - \mu H}}} \right) \tan \left( \frac{\pi}{2} \sqrt{\frac{\nu - \mu H}{1 + \nu - \mu H}} \right) \right) \\
&\leq a_n \left( \frac{\rho d}{2} |J|^2 - \frac{\alpha}{2} |J| + \frac{2|J|^2}{\pi^2} \left( \frac{\sqrt{3}\pi}{4} \right) \tan \left( \frac{\pi}{2} \sqrt{\frac{\nu - \mu H}{1 + \nu - \mu H}} \right) \right),
\end{aligned}$$

where, in deriving the last inequality, we have used the fact that  $C_v^s \leq 1$  and  $\nu - \mu H > \frac{1}{2}$ .  
By using the inequality

$$\tan \left( \frac{\pi}{2} \sqrt{\frac{x}{1+x}} \right) < 1 + \frac{4}{\pi} x, \quad \forall x > 0,$$

and

$$\nu - \mu H = \frac{1}{2} \alpha |J| + \frac{1}{2} + \frac{Hd}{2} (1 - C_v^s),$$

then we obtain

$$\begin{aligned}
C_n^1 &\leq a_n \left( \frac{\rho d}{2} |J|^2 - \frac{\alpha}{2} |J| + \frac{2|J|^2}{\pi^2} \left( \frac{\sqrt{3}\pi}{4} \right) \left( 1 + \frac{4}{\pi} (\nu - \mu H) \right) \right) \\
&= \frac{a_n |J|}{2} (a |J|^2 + b |J| - c),
\end{aligned}$$

where

$$a = \frac{2\sqrt{3}\alpha}{\pi^2},$$

$$b = \rho d + \frac{\sqrt{3}}{\pi} + \frac{2\sqrt{3}}{\pi^2} [1 + Hd(1 - C_v^s)],$$

$$c = \alpha.$$

Let

$$J_0^* = \frac{-b + \sqrt{b^2 + 4ac}}{2a},$$

then  $C_n^1 \leq 0$ ,  $\forall n \geq 0$  when  $0 \leq |J| \leq J_0^*$ . Hence from (3.12),  $T'(t) \leq 0$  for  $t > 0$ . I

**Remark 3.3** Corresponding to the example given in the last section, with the data given in Table 1, we have

$$J_0^* \approx 0.9780.$$

Our computation (see Fig. 2) confirms that when  $|J| < J_0^*$ ,  $T(t)$  is always decreasing, and for some values  $|J| > J_0^*$ ,  $T(t)$  decreases first and then increases to its steady state  $T_\infty$ .

**Remark 3.4** The results in Theorem 3.2 cease to hold whenever  $C_v^s$  is a function of temperature, which is the case during the phase transformation of SMA. Even though we have not proved such a theorem yet for the variable  $C_v^s$  case, a similar result should be valid for the variable  $C_v^s$  case. In fact, the constant  $C_v^s$  case is an approximation of the variable  $C_v^s$  case, hence the formula for the value  $J_0^*$  may be used in the variable  $C_v^s$  case approximately by replacing  $C_v^s$  by an average  $\langle C_v^s \rangle$ . This is numerically confirmed by the example given in this paper, as shown in Fig. 3, for  $C_v^s(T)$  given by (2.19) and (2.20).

## 4 CONCLUSION

In this paper we studied the solution behavior of the transient heat transfer problem of the 1-D symmetric thermoelectric shape memory alloy (SMA) actuator. From the analytical model proposed in [13], we proved that for the transient cooling problem with constant electric current density of magnitude  $|J|$  there is a value  $J_0$  of  $|J|$  such that when  $|J| < J_0$ , the temperature in SMA is always decreasing to its steady state, and when  $|J| > J_0$ , the temperature in SMA may not be always decreasing, which is an important property of thermoelectric SMA actuators. A lower bound for the value  $J_0$  was obtained.

Due to the relatively large heat conduction coefficient of the SMA compared with the heat conduction coefficient of the P-type and N-type semiconductors, the temperature distribution in the SMA material is expected to be almost constant for small ratio  $d/L$ , and the results derived in this paper are valid. When the ratio  $d/L$  is not small and the physical domain of an SMA actuator is no longer 1-D, a proper three dimensional model of SMA actuators should be introduced and studied.

**Acknowledgement:** This work was supported by the Grant No. N00014-94-1-0268 from the office of Naval Research to Texas A&M University, College Station. The authors acknowledge many useful discussions with Professor E. Straube and Dr. A. Bhattacharyya.

## Appendix

**Lemma A.1** *Let  $J(t) = J < 0$ . Let  $\hat{w}(s)$  be the transfer function of equation (2.14). Then there exists a sequence of positive numbers  $\{a_n\}_{n=0}^{\infty}$  such that*

$$\hat{w}(s) = \frac{1}{\hat{G}(s)(s+H) + \mu s + \nu} = \sum_{n=0}^{\infty} \frac{a_n}{s - \lambda_n}, \quad s \in \Re, \quad (\text{A.1})$$

where  $\{\lambda_n\}_{n=0}^{\infty}$  are the single poles of  $\hat{w}(s)$ .

**Proof.** From the proof of Theorem 3.1, we have

$$\cdots < \lambda_n < \lambda_{n-1} < \cdots < \lambda_1 < \lambda_0 < -H, \quad \text{and} \quad \lambda_n \simeq O(n^2),$$

because  $J < 0$ . Generalize the transfer function  $\hat{w}(s)$  to the complex plane  $\mathbb{C}$ , i.e.  $\hat{w}(z)$ ,  $z \in \mathbb{C}$ . Thus  $\hat{w}(z)$  is analytic everywhere except  $\{\lambda_n\}$  and each  $\lambda_n$  is a simple pole of  $\hat{w}(z)$ . The residue of  $\hat{w}(z)$  at  $z = \lambda_n$  is given by

$$\text{Res}(\hat{w}(z), \lambda_n) = \left[ \sum_{m=1}^{\infty} \frac{v_m - H}{(\lambda_n + v_m)^2} + \mu \right]^{-1} \leq \frac{1}{\mu}.$$

Let  $a_n = \text{Res}(\hat{w}(z), \lambda_n)$ . Then  $a_n \geq 0$  and  $\sum_{n=0}^{\infty} \frac{a_n}{z - \lambda_n}$  is convergent for any  $z \in \mathbb{C} \setminus \{\lambda_n\}$  because  $\lambda_n \simeq O(n^2)$ . Let

$$f(z) = \hat{w}(z) - \sum_{n=0}^{\infty} \frac{a_n}{z - \lambda_n}.$$

Since  $f(z)$  has no poles in  $\mathbb{C}$  and  $\hat{w}(z)$  is analytic on  $\mathbb{C} \setminus \{\lambda_n\}$ ,  $f(z)$  is an entire function [7]. We will prove that  $f(z) = 0$ ,  $z \in \mathbb{C}$ .

Let  $z = (v_m - H)e^{i\theta} - H$ ,  $\theta \in [0, 2\pi]$  and  $m$  be sufficiently large. We have

$$|\hat{w}(z)| = \frac{1}{|\hat{G}(z)(z+H) + \mu z + \nu|}$$

$$\begin{aligned}
&= \frac{1}{\left| \sum_{n \neq m} \frac{m^2 e^{i\theta}}{m^2 e^{i\theta} + n^2} + \frac{e^{i\theta}}{e^{i\theta} + 1} + (\nu - \mu H) + \mu m^2 \pi^2 e^{i\theta} \right|} \\
&= \frac{1}{\left| \sum_{n \neq m} \frac{m^2}{m^2 e^{i\theta} + n^2} + \frac{1}{e^{i\theta} + 1} + (\nu - \mu H) e^{-i\theta} + \mu m^2 \pi^2 \right|} \\
&\leq \frac{1}{\left| \frac{1}{e^{i\theta} + 1} + \mu m^2 \pi^2 \right| - (\nu - \mu H) - \sum_{n \neq m} \frac{m^2}{|n^2 - m^2|}}.
\end{aligned}$$

By the inequality  $|a + bi| \geq |a|$  for any  $a, b \in \mathfrak{R}$ , we obtain

$$\left| \frac{1}{e^{i\theta} + 1} + \frac{\mu K \pi^2}{C_v L^2} m^2 \right| \geq \frac{1}{2} + \mu m^2 \pi^2.$$

Note that

$$\begin{aligned}
&\sum_{n \neq m} \frac{m^2}{|n^2 - m^2|} \\
&\leq \int_0^{m-1} \frac{m^2}{m^2 - x^2} dx + \frac{m^2}{m^2 - (m-1)^2} + \frac{m^2}{(m+1)^2 - m^2} + \int_{m+1}^{\infty} \frac{m^2}{x^2 - m^2} dx \\
&= \frac{4m^3}{2m^2 - 1} + \frac{m}{2} \ln \frac{m+x}{m-x} \Big|_0^{m-1} + \frac{m}{2} \ln \frac{x-m}{x+m} \Big|_{m+1}^{\infty} \\
&= \frac{4m^3}{2m^2 - 1} + \frac{m}{2} \ln(2m-1) - \frac{m}{2} \ln(2m+1) \\
&\leq 2m + 2.
\end{aligned}$$

We hence obtain

$$|\hat{w}(z)| \leq \frac{1}{\frac{1}{2} + \mu m^2 \pi^2 - (\nu - \mu H) - (2m + 2)}.$$

Also for  $z = (v_m - H)e^{i\theta} - H$ ,  $\theta \in [0, 2\pi]$  and  $m \geq 2$ , by using the properties of  $\{\lambda_n\}$  and the expressions of  $\{a_n\}$ , we have

$$\left| \sum_{n=0}^{\infty} \frac{a_n}{z - \lambda_n} \right| \leq \sum_{n=0}^{\infty} \frac{a_n}{|(v_m - H)e^{i\theta} - (\lambda_n + H)|}$$

$$\begin{aligned}
&\leq \sum_{n=0}^{\infty} \frac{a_n}{|v_m + \lambda_n|} \\
&\leq \sum_{n=1}^{m-1} \frac{a_{n-1}}{v_m - v_n} + \frac{a_{m-1}}{|v_m + \lambda_{m-1}|} + \frac{a_m}{|v_m + \lambda_m|} + \sum_{n=m+1}^{\infty} \frac{a_n}{v_n - v_m} \\
&\leq \frac{1}{\mu\pi^2} \sum_{n \neq m} \frac{1}{|n^2 - m^2|} + \frac{v_{m+1} - v_{m-1}}{v_m - H} \\
&\leq \frac{1}{\mu\pi^2} \left( \frac{2m+2}{m^2} \right) + \frac{4}{m}.
\end{aligned}$$

Therefore for  $z = (v_m - H)e^{i\theta} - H$ ,  $\theta \in [0, 2\pi]$  and  $m$  be sufficiently large, we have

$$\begin{aligned}
|f(z)| &\leq |\hat{w}(z)| + \left| \sum_{n=0}^{\infty} \frac{a_n}{z - \lambda_n} \right| \\
&\leq \frac{1}{\frac{1}{2} + \mu m^2 \pi^2 - (\nu - \mu H) - (2m+2)} + \frac{1}{\mu\pi^2} \frac{2m+2}{m^2} + \frac{4}{m}.
\end{aligned}$$

Since  $f(z)$  is an entire function, by applying the Maximum Modulus Theorem [7], we obtain that  $f(z) = 0$  on  $\mathbb{C}$ . Hence (A.1) is proved.  $\blacksquare$

By the same way, the following lemma can also be derived.

**Lemma A.2** *Let  $J(t) = J < 0$ . Then there exists a sequence of real numbers  $\{b_n\}_{n=0}^{\infty}$  such that*

$$\frac{\mu + \hat{G}(s)}{\hat{G}(s)(s + H) + \mu s + \nu} = \sum_{n=0}^{\infty} \frac{b_n}{s - \lambda_n}, \quad s \in \mathfrak{R}. \quad (\text{A.2})$$

Furthermore,

$$b_n = (\hat{G}(\lambda_n) + \mu)a_n, \quad \text{and} \quad \sum_{n=0}^{\infty} b_n = 1. \quad (\text{A.3})$$

## 5 REFERENCES

1. Bhattacharyya A., D.C. Lagoudas, Y. Wang and V. K. Kinra, "Thermoelectrical cooling of shape memory alloy actuators: theoretical modelling and experiment", *Active Materials and Smart Structures*, ed. by G. L. Anderson and D. C. Lagoudas, SPIE Vol. 2427(1994), pp. 198-217

2. Boyd, J.G., and D. C. Lagoudas, "Thermomechanical Response of Shape Memory Composites", *Journal of Intelligent Materials and Structures*, 5(1994), pp. 333-346.
3. Boyd, J.G., and D. C. Lagoudas, "A thermodynamical constitutive model for shape memory materials. Part I. The monolithic shape memory alloy", to appear in *Int. J. of Plasticity*, 1995.
4. Buckingham, E., "On physically similar system illustrations of the use of dynamical equations", *Phys. Rev.*, 4(1914), pp 345-376.
5. Buehler W. J. and R. C. Wiley , "Nickel-base alloys", *US patent* 3174(1965).
6. Carslaw, H. S. and J. C. Jaeger, *Conduction of Heat in Solids*, 2nd edition, Clarendon Press, Oxford, 1959.
7. Conway, J. B. *Functions of One Complex Variable*, 2nd edition, Springer-Verlag, New York, 1988.
8. Domenciali, C. A., "Irreversible thermodynamics of thermoelectricity", *Reviews of Modern Physics*, Vol. 26(1954), pp. 237-275.
9. Edelen, D. G., "On the characterization of fluxes in nonlinear irreversible thermodynamics", *Int. J. Eng. Sci.*, 12(1974), pp. 397-411.
10. Goebel, K. and W. A. Kirk, *Topics in metric fixed point theory*, Cambridge University Press, New York, 1990.
11. Hochstadt, H., *Integral Equations*, John Wiley & Sons Inc., New York, 1973.
12. Jackson, C.M., Wagner, H.J., and Wasilewski, *A report on 55-Nitinol – the alloy with a memory: Its Physical Metallurgy, Properties and Applications*, Technology Utilization Office, NASA, Washington, D.C., 1972.
13. Lagoudas, D. C. and Z. Ding, *Modelling of thermoelectric heat transfer in Shape Memory Alloy actuators: transient and multiple cycle solutions*, to appear in *Int. J. Engrg. Sci.*, 1995.
14. Tanaka, K. and S. Nagaki, "A thermomechanical description of materials with internal variables in the process of phase transformations", *Ing. Arch.* 51(1982), pp. 287-299.